Research Project Paper

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OUTLINE

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Statement of the Problem

The following is an excerpt of Article I, Section 2 of the Constitution of the United States on apportionment[1]:

> *"…Representatives shall be apportioned among the several states according to their respective numbers, counting the whole number of persons in each State…each State shall have at Least one Representative."*

If we could give each state fractional numbers of representatives, then apportionment would have a natural solution, namely its quota, where quota is equal to the fraction of each state population over the total population multiplied by the number of representatives. Since we cannot have fractions of representatives, each state must be given an integral number of representatives approximating its quota. But the question is how?

Consider, for example, a country with five states and populations as given in Table 1. The exact proportional solutions are given for houses of 25, 26, and 27 seats. For a house of 25 seats a reasonable solution is 9, 7, 5, 3, 1 seats for A, B, C, D, and E, respectively. But, in a house of 26 or 27 seats how should the representatives be assigned?

State	Population	25 Seats Ex.Quota	26 Seats Ex.Quota	27 Seats Ex.Quota
A	9061	8.713	9.061	9.410
B	7179	6.903	7.179	7.455
C	5259	5.057	5.259	5.461
D	3319	3.191	3.319	3.447
E	1182	1.137	1.182	1.227
	26000	25	26	27

Table 1.
(Fair representation, Balinski, M.L., and Young, H. P., P. 702)

[1] According to Daniel Webster: To apportion is to distribute by the right measure, to set off in just parts, to assign in due and proper proportion.

Here are some formal notations, used throughout this paper:

- **P =** the total population
- **h** = the house size (number of representatives)
- **p_i** = the population of state i
- **q_i** = the exact quota for state i
- **a_i** = the assigned number of representatives to state i

The exact quota for state i, **q_i**, is calculated by the following formula:

$$q_i = (p_i / P) * h$$

The Hamilton Method

The very first method that was suggested by Hamilton is very intuitive. Given that there are **s** states with populations $(\mathbf{p}_1, \mathbf{p}_{2, \ldots,} \mathbf{p}_s)$ here is how this method works:

1? Calculate the exact quota for all the states.
Meaning: $\mathbf{q}_i = (\mathbf{p}_i / \mathbf{P}) * \mathbf{h}$ (for all $1 \leq i \leq s$)

2? Each state is temporarily assigned the integral part of its exact quota.
Meaning: $\overline{\mathbf{a}}_i = \lfloor \mathbf{q}_i \rfloor$ (for all $1 \leq i \leq s$)

3? The fraction parts of these derived numbers are sorted from largest to smallest.

4? The total number of representatives that are already assigned, **å `a$_{i,}$** is deducted from **h**. As long as at least one representative is left to be given, start from the biggest fraction and give each state in the priority list another representative. This means that the first $\mathbf{h} - \sum \overline{\mathbf{a}}_i$ states in the list receive an extra representative. (If there are ties between two of these fractions, then there exists distinct arrangements of the priority list each of which leads to a solution of the given problem.)

However, the solution this method provides may result in what is known as *the Alabama paradox:* If the number of representatives increases by one, a state can lose a representative.

This problem first happened to Alabama in 1880 as shown in Table 2.

State	Quota at 299	Allotment	Quota at 300	Allotment
Alabama	7.646	8	7.671	7
Texas	9.640	9	9.672	10
Illinois	18.640	18	18.702	19

Table 2.
The Alabama Paradox after the 1880 Census
(Democracy delayed: Congressional reapportionment and urban-rural conflict in the 1920s, University of Georgia Press, P. 29)

Now let's look at another example of the Hamilton method which also results in the Alabama Paradox. Consider, for example, a country with four states and populations as given in Table 3. The exact proportional solutions for houses of 20 and 21 seats.

State	Population	20 seats Ex. Quota	Allotment	21 seats Ex. Quota	Allotment
A	45	10.35	10	10.86	11
B	13	2.99	3	3.14	3
C	27	6.20	6	6.52	7
D	2	0.46	1	0.48	0
	87	20	20	21	21

Table 3.

According to the Hamilton method, for a house of 20 seats the apportionment is 10, 3, 6 and 1 for states A, B, C and D respectively. Finally for a house of 21 seats the apportionment is 11, 3, 7, and 0. As it is observed when the number of representatives increases from 20 to 21, state D loses a representative instead of gaining one.

Given the argument above, the followings are desirable properties for a given method.

1? If everything remains constant, and **h** is increased by one, no state should lose a representative. (Avoiding the Alabama Paradox)

2? Satisfy the quota: $\lfloor \mathbf{q_i} \rfloor$ **£** $\mathbf{a_i}$ **£** $\lceil \mathbf{q_i} \rceil$ (for all $1 \le i \le s$)

The Divisor Schemes

Many apportionment techniques have been suggested and presented. The following is the list of five of these suggested methods, which are all divisor methods. They all avoid the Alabama Paradox.

- Smallest Divisors (SD)
- Harmonic Mean (HM)
- Equal Proportions (EP)
- Webster (Web) - also known as Major Fractions
- Jefferson (Jeff) - also known as Greatest Divisors or d'Hondt

A divisor method works in the following way:

1? The population of each state is divided by a function of its already assigned number of representatives, **d(`a$_i$)**.

2? These numbers are sorted from largest to smallest in a priority list. The state with the highest ranking gets one representative, and **h** is decremented by one.

3? Steps 1 and 2 are repeated recursively until there are no more representatives to be assigned.

Table 4 shows the list of all five methods along with their divisor criterion and rank-indices.

Method	Rank-Index	Divisor Criterion d(a)
Smallest Divisors	P/a	A
Harmonic Mean	P/[2(a+1)/(2a+1)]	2(a+1)/(2a+1)
Equal Proportions	P/[a(a+1)]$^{1/2}$	a(a+1)$^{1/2}$
Webster	P/(a+1/2)	a+1/2
Jefferson	P/(a+1)	a+1

Table 4.

In the methods listed above, some favor large states and others favor small states when it comes to assigning representatives. And the methods are listed in Table 4, in order of increasing favoritism to large states, with Jefferson most favoring large states at one end, and SD favoring small states at the other. Formally:

> Consider two states with populations: p^* and $p-$ where $p^* > p-$. Suppose that a solution $f' \in M'$, gives a^* many representatives to p^*, and $a-$ many representatives to $p-$ at some house size of h'. And that $f'' \in M''$ accords a total of $a^* + a-$ to these two states at some house size of h''. Then f'' favors the large state over f' if it gives at least a^* seats to p^* at h''. According to Balinski and Young, for any such choices of p^*, $p-$, h' and h'', a method M'' **favors large states over** M' if any solution $f'' \in M''$ favors the large state over any $f' \in M'$.

Our Suggested Recursive Schemes

We suggest a new recursive scheme that can be applied to any of the above solutions. Here is a description of how this scheme works, given that we have **s** states:

1? Choose a method **M** of assigning representatives.

2? Assign representatives according to **M.**

3? Select a state i, based on a selection-criterion, and let that state walk away with its assigned representatives.

4? Subtract $\mathbf{p_i}$ from **P** and $\mathbf{a_i}$ from **h**.

5? Go back to step 2 unless there are no more representatives to be assigned.

A selection-criterion can be any of the following:

- **REC1 -** The least absolute error – compute $|\mathbf{a_i} - \mathbf{q_i}|$ for all the states, sort the results and select the state with the least difference.
- **REC2 -** The least relative error – compute $|(\mathbf{q_i} - \mathbf{a_i}) / \mathbf{q_i}|$ for all the states, sort the results and select the state with the smallest relative error.

Applying these recursive schemes to the five methods discussed before yields: eighteen solutions that we have implemented and can be found in the attachments.

Test Cases
&

Implemented Schemes

Implemented Schemes

1? SD
2? HM
3? EP
4? Web
5? Jeff
6? Hamilton
7? Rec1 - (with Hamilton method)
8? Rec2 - (with Hamilton method)
9? SDRec1
10? SDRec2
11? HMRec1
12? HMRec2
13? EPRec1
14? EPRec2
15? WebRec1
16? WebRec2
17? JeffRec1
18? JeffRec2

- Those methods that are underlined in the charts have produced the Alabama Paradox
- The Quota method's (Q) results were taken from the Balinski and Young book. (Refer to references).

Conclusion

Based on our test cases and the desirable properties mentioned earlier here is our conclusion:

- Applying Rec1 and Rec2 to the Hamilton method did not avoid the Alabama Paradox.

- When combining Rec1 and Rec2 with the five divisor methods, the results are the same as the non recursive versions.

- Looking at the charts, we picked Rec2 (combined with Hamilton method) as a good scheme based on our desired set of properties.

Open for Further Discussion

Balinski and Young, have suggested another method of apportionment called the Quota method. The Quota method is the only method that not only avoids the Alabama Paradox, but it also satisfies quota, a condition we had mentioned earlier in our desired set of properties. We have included a table at the end of the paper, which supports this argument.

One of the desirable properties we thought of but did not investigate in this paper is the following:

- If $\mathbf{p_i} / \mathbf{P}$ remains constant for all i's, and $\mathbf{h}$ stays the same too, no state should lose a representative.

For example, if the populations of all the states doubled and the number of representatives remained the same, is it fair for a state to lose a representative?

References

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