Generalized Ramsey Numbers for Cycles

The Algorithm

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1 What our goals are

This paper is an exposition of the paper "Generalized and geometric Ramsey numbers for cycles." *preprint*, 1998. by Károlyi, Gyula and Rosta, Vera. Our (unrealized) goal was to come up with a constructive proof of this theorem.

2 Main Theorem

Let C_k denote the cycle of length k. The generalized Ramsey number of the pair (C_k, C_k), denoted by $R(C_k, C_k)$, or $R(C_k)$ for brevity, is the number of vertices of the smallest possible complete graph such that, when its edges are randomly colored red and blue, the graph is guaranteed to contain a cycle of size k (a C_k) that is red, or a cycle of size k that is blue.

Theorem 1 Let $k \geq 3$, where k is an integer.

$$R(C_k) = \begin{cases} 6 & \text{if k = 3 or 4,} \\ \frac{3}{2}k - 1 & \text{if k is even,} \\ 2k - 1 & \text{otherwise (if k is odd).} \end{cases} \tag{1}$$

Intuition You desire a complete graph to have a red cycle of size k or a blue cycle of size k, regardless of how the edges are actually colored. (The names of the colors are arbitrary, of course, and reversible.)

3 Lemmas

3.1 Lemma 0

Let G be any complete graph whose edges are colored red or blue.

1. Suppose that G contains a monochromatic C_{2l+1} for some $l \geq 3$. Then G also contains a monochromatic C_{2l}. The point to note here is that $2l$ guarantees the length of the cycle is even, and $2l + 1$ guarantees that it is odd. Therefore, if a graph contains an odd-length monochromatic cycle, there is also an even-length monochromatic cycle of one less. Also note that the two cycles are not necessarily the same color.

2. Suppose that G contains a monochromatic C_{2l} for some $l \geq 3$. Then G also contains a monochromatic C_{2l-2}. The point to note here is that $2l$ is still even, and so is $2l - 2$. This allows us to reapply this lemma to show that if a graph contains an even-length monochromatic cycle, it also contains all even length monochromatic cycles less than $2l$ but greater than or equal to 6 (since $l \geq 3$). Also note that the two cycles are not necessarily the same color.

3. Suppose that G contains a (monochromatic) blue cycle $C = x_1 x_2 \cdots x_{2l} x_1$, but does not contain any monochromatic C_{2l-1}. (Unlike point 1, the existence of an even cycle does not imply the existence of an odd cycle of length one less) Then each of the complete subgraphs G_1 and G_2, induced on the vertex sets $\{x_1, x_3, \ldots, x_{2l-1}\}$ and $\{x_2, x_4, \ldots, x_{2l}\}$ respectively, is a red K_l.

4. Suppose that G contains a blue cycle $C = x_1 x_2 \cdots x_{2l} x_1$ such that $G_1 = \{x_1, x_3, \ldots, x_{2l-1}\}$ and $G_2 = \{x_2, x_4, \ldots, x_{2l}\}$ are red complete subgraphs with l vertices (as per point 3). Then one of the following 3 possibilities occurs:

   (a) G contains a red C_m for each $3 \leq m \leq 2l$.

(b) G contains a blue C_r for each $3 < r \leq 2l+1$ resp. for each $3 \leq r \leq 2l+1$, according to the cases $l \geq 3$ and $l = 2$, respectively.

(c) G contains a blue C_r for each even number $4 \leq r \leq 2l$ and a red C_m for each $3 \leq m \leq \min\{\lceil |V(G)|/2 \rceil, 2l\}$.

3.2 Lemma 1

Let G be a complete graph with $\frac{3}{2}k$ vertices, where $k > 4$. If we color the edges of G with red and blue, then either there exists a monochromatic cycle of length at least k or there is a blue C_k in G. In particular, if $m > 4$ is an even number, then any complete graph with $\frac{3}{2}m - 1$ vertices, whose edges are colored with two colors, contains a monochromatic cycle of at least length m.

Intuition The first part of this lemma covers the case with no restriction on the parity of k. The second part of the lemma allows us to actually nail down that there is a cycle of length k, without having to worry about there being a blue C_k, by plugging in k or $k+1$ for m (to make it even as necessary). Both of these results will be utilized in Lemma 2.

3.2.1 Lemma 1.1

Let $P = u_1 x_1 u_2 \cdots u_{s-1} x_t u_s$ be a maximal alternating path. Then either $s = \frac{k}{2} - 1$ or $L - 5 \leq t \leq L - 3$. Moreover, if $s < \frac{k}{2} - 1$ and $t = L - 5$, then $x_{t-1} \in P$.

3.2.2 Lemma 1.2

Let $P = u_1 x_1 u_2 \cdots u_{s-1} x_t u_s$ be a maximal alternating path. If $t \leq L - 4$ then P can be altered to a blue C_{2s+2}. Similarly, if $t = L - 3$, then P can be altered to either a blue C_{2s+2} or to a blue C_{2s+1}.

Intuition The whole point of this lemma is to show that since we have this maximal alternating path, we can alter it to be a blue cycle of length $2s + 2 = k$. Property (ii) shows that $t \leq 2s - 3$. We can show that $t < L - 5$. (The proof says $t \leq L - 5$, but I think this is an error.) If $t = L - 3$ or $t = L - 4$, then we would have $L \leq 2s$ and $L \leq 2s + 1$ respectively, which is contradictory to no cycle being longer than L. Also, if $t = L - 5$ and $x_{t-1} \in P$, then property (ii) implies that $t - 1 \leq 2(s-1) - 3$, that is $L \leq 2s + 1$, and again this is contradictory. Therefore, $t < L - 5$, and then from Lemma 1.1, $s = \frac{n}{2} - 1$. Therefore, given Lemma 1.2, we can alter P to be a blue cycle of length $2s + 2$ which is equal to k, and we are done.

3.3 Lemma 2

Let G be any complete graph with (at least) as many vertices as it is indicated in Theorem 1. If we color the edges of G with red and blue, then G contains a monochromatic C_k.

4 Algorithm

Let G be a graph with as many vertices n as indicated by the theorem for input k.

1. Using a depth-first search on the graph on only the red edges, find the longest cycle C in G, and let $L \leq n - 1$ denote the number of vertices. Depth first search's running time is $\Theta(V + E)$, where V is the number of vertices and E the number of edges. Since we can only be sure the number of red edges will be at most $\binom{n}{2} \leq n^2$, the running time for this part is $O(n^2 + n)$. *I'm still not satisfied that this part, finding the longest cycle, isn't computationally intensive. We know a cycle exists, but the paper doesn't really talk about this part. More and more I think this may be the non-constructive part of the paper, and the rest of the paper builds off of knowing that this cycle exists, but not necessarily any smooth way of finding it. Maybe knowing it exists is enough to ensure the complexity isn't too bad. Maybe DFS isn't the best way to do it.*

2. If $L = k$, done immediately. If $L > k$, then continue below to find a blue alternating path. If $L < k$, then restart by finding the longest monochromatic *blue* cycle, and in the steps below switch the roles of red and blue. The main theorem tells us a cycle exists of size k; therefore if one of at least that size does not exist that is red, then there must be one that is blue.

3. Let $D = V(G) - V(C)$. $|D| \geq \frac{k}{2}$. Find a maximal *blue alternating path*.

   Recall that an alternating path $P = u_1 x_1 u_2 \cdots u_{s-1} x_t u_s$, where $x_i \in V(C)$ and $u_i \in D$.

   To do this, begin with an alternating path $P = u_1 x_1 u_2$, and then maximize it with the following sub-algorithm. Since we let $z_1 = x_1$, $P = u_1 x_1 u_2 \cdots u_{s-1} x_t u_s$, where $t \leq L - 3$.

   (a) Pick $y_1, y_2 \in D - P$; two non-cycle vertices that are also not in the path.

   (b) If $u_s x_{t+m} y_j$ is a blue path for $1 \leq j, m \leq 2$, then add y_j to the path, and restart this sub-algorithm with this new path.

   (c) When there are no more y_1, y_2 pairs to be found, you have the maximal path.

   The time analysis of this sub-algorithm, in the worst case, is not too bad. We don't have to consider all possible pairs y_1, y_2 in the first each time we do it; we just find any pair that works and go with it. Our repeated restarting of the algorithm will eventually lead us to cover all those pairs. So our worst-case running time is $O(n)$.

4. We can now take our maximal alternating path P and alter it to a cycle of at least as large as we need as follows. These steps are all $O(1)$.

   Let $u \in D - P$ and $j \in \{1, s\}$. Let Q_j denote the blue path $u_j x_i u$ for some $t + 1 \leq i \leq L$ if it exists, or the blue "path" $u_j u$ otherwise. u_j will either be the first or last vertex in the alternating path, and x_i will be a cycle vertex not covered by the path. Let Q'_j denote the path obtained from Q_j by reversing the order of its vertices. Q_j will either be of length 2 or of length 3, depending on whether or not the blue path $u_j x_i u$ exists.

   (a) If $|Q_1| = |Q_s| = 3$, and $|Q_1 \cup Q_s| = 5$, then $PQ_sQ'_1$ is a blue C_{2s+2}. If either $|Q_1| = 3$, $|Q_s| = 2$ or $|Q_1| = 2$, $|Q_s| = 3$, then $t = L - 3$ and $PQ_sQ'_1$ is a blue C_{2s+1}.

   (b) If $|Q_1| = |Q_s| = 2$, then $t = L - 3$ and the path $u_s u u_1$ is blue. Let P_{-i} denote the path obtained from P by deleting its last i vertices and the corresponding edges. Either $P_{-1} x_{t+2} u_s u u_1$ is a blue C_{2s+1}, or $P_{-1} x_{t+2} u u_s x_{t+3} u_1$ is a blue C_{2s+2}.

   (c) If $|Q_1| = |Q_s| = 3$ and $|Q_1 \cup Q_s| = 4$, then you have to use similar arguments I haven't worked out yet.

5. At this point, we have a large cycle, but not necessarily one of the desired length. Now we will reduce it to the desired length. Let a *j − chord* denote the edge of the form $x_i x_{i+j}$. All these cases come directly out of Lemma 0.

   (a) If the cycle we have is of odd length, we can obtain a cycle of even length. If there is a blue 2-chord on the original cycle, we can use this 2-chord to form a cycle of length one less. If all 2-chords are red, then somewhere there is a 3-chord that is red that will allow us to form a red cycle of length one less.

   (b) If the cycle we have is of even length, we can obtain a cycle of length two less (that is still even). If there are any blue 2-chords, then we have this cycle by applying (a) to this intermediate cycle we form with the blue 2-chord. Otherwise, all 2-chords are red, and we can find a red 3-chord to give us a red cycle of the desired length.

   (c) If the cycle we have is of even length, but there is no cycle of length one less, then each of the complete subgraphs G_1, G_2 induced on the vertex sets $\{x_1, x_3, \ldots, x_{2l-1}\}$ and $\{x_2, x_4, \ldots, x_{2l}\}$ is a red K_l.

   (d) If there are two independent red edges connecting G_1 and G_2, then you can construct a red C_n.

(e) Otherwise, all the edges are blue with the possible exception of some edges that have a common vertex x. Since $x \in C$, find the two blue edges connecting x to the other subgraph and construct a blue C_k for any even $2 < k \leq 2l$.

(f) Otherwise, let $D = V(G) - V(C)$. If a vertex in D is connected to G_1 and G_2 by some blue edges, then you can use them to create a blue C_k for any $3 < k \leq 2l + 1$. Otherwise, all these edges are red, and you can use them to find a red C_k.

5 Bibliography

Károlyi, Gyula and Rosta, Vera. “Generalized and geometric Ramsey numbers for cycles.” *preprint*, 1998.

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