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Methods of Apportionment

Apportionment of Representatives in the United States Congress House of Representatives and avoiding the 'Alabama Paradox'

Introduction

The problem of how many representatives should be allotted to a state existed since the beginning of the Republic. The Constitution dictates that each state shall be represented in proportion to its population (Article I Section 2, "Representatives… shall be apportioned among the several States, which may be included within this Union, according to their respective numbers…"). The constitution does not specify how the apportionment is to be worked out, that is the problem. In the United States, Congress decides how many representatives will comprise the House of Representatives and how many representatives each state shall have. This sounds like a straightforward mathematical problem of applying the idea of one man, one vote. But congressmen are human beings (Really!) and a state cannot be assigned one and two fifths of a representative.

The intent of the Constitution was captured by Daniel Webster, who said "To *apportion* is to distribute by right measure, to set off in just parts, to assign in due and proper proportion." But we can't assign fractions of representatives, and Webster realized that "that which cannot be done perfectly must be done in a manner as near perfection as can be." Debates, reports, methods and bills, have succeeded from the beginning of the republic until our time, following each census.

Desired properties for State Representation methods

Methods should have as many qualities from the list below as possible. A perfect method would of course satisfy all of them.

1. No state's number of representatives should decrease, if the total number of Representatives increases.
2. Every state should have within one (exclusive) of their quotient. For example if a state should receive 3.4 representatives it can receive 3 or 4. If the state should receive exactly 3 representatives, it should receive 3, but not 2 or 4. In future, we will call having this property -satisfying Quota.
3. All states abide by the same formula for representation
4. Methods do not artificially favor large states at the expense of the smaller ones and vice versa.

In addition Constitution requires US House of Representatives representation to have the following property(s).

5. Every State should have at least one representative

Methods

Hamilton method

Many of the Founding fathers came up with their own methods for apportionment.
Treasury Secretary Hamilton proposed to compute the exact proportionate number for each state, and then reduce each to a whole number. (Remember that fractions are not allowed.) The total will almost always be less than actual number of representatives in the congress. Remaining seats will be distributed one by one by assigning an "extra" representative to the state that had the largest fraction dropped, the second largest etc. Hamilton's method satisfies all the rules, except rule number one. The fact that a state can lose a representative even though the size of the congress has increased is called the *Alabama Paradox.* It can be shown that Hamilton method does have the Alabama paradox (named after the state that was first the victim of this phenomenon). For example, assume we have four states with the populations given in the table below with 68 representatives in the House.

STATE NAME	Population	EXACT REPRESENTATION	Representatives
A	230	1.565	2
B	600	4.082	4
C	4320	29.388	29
D	4850	32.993	33

When the number of Representatives increases to 69, one would expect state C to receive a representative, since it has biggest fraction cut off (0.388). But C and D each gain a representative and state A loses one. C has an exact fraction, and new fractional part of the exact representation for state D is higher than state A's. State A actually loses a representative to D.

STATE NAME	Population	EXACT REPRESENTATION	Representatives
A	230	1.597	1
B	600	4.167	4

C	4320	30.000	30
D	4850	33.681	34

In 1792 Congress passed the first act of apportionment, setting the number of representatives to 120 from 15 states in the Union and approved Hamilton's method as a method for apportionment. But US President George Washington vetoed it; it was his first veto and one of only two that he exercised in eight years of being the President. There is no evidence that the reason Washington vetoed the method was the Alabama paradox; there is no evidence that he even knew about it. George Washington favored another method, put forth by the Secretary of the State Thomas Jefferson. Jefferson did not like Hamilton's method and commented that "No invasion of the Constitution are fundamentally so dangerous as the tricks played on their own numbers, apportionment."

Jefferson's Method

Jefferson came up with what is known as the method of greatest divisors. Suppose we are given state populations p1, p2, .., pN and representative apportionment a1, a2, .., pN. We can calculate a divisor L(s) = a(s)+1 for each state s. Then states can be ranked using the p(s)/L(s) ratios. The higher this ratio, the more deserving this state is to get another representative.

Everybody starts with zero representatives. The representatives are always assigned to the state with the current highest ratio (rank-index). The first N representatives are assigned one to each state. This is naturally enforcing US Constitution rule about each state having a minimum of one representative. The divisor choice L(s) = a(s)+1 is natural, because it ranks how much better off the state will be if it was given one more representative. The divisor choice Jefferson's Method uses is arbitrary, since other methods use divisors such as L(s) = a(s) + ½, L(s) = a(s) or L(s) = √(a(s) * (a(s)+1)).

Jefferson never mentioned favoring large states as a flaw of his method, but he probably knew it. Both, Jefferson and Washington were from a large state, Virginia to be precise. It happens that in the first apportionment, in 1792, the largest state, Virginia (pop. 630,558) was awarded an additional representative, at the expense of the smallest state, Delaware (pop. 55,538). Jefferson's method was followed more or less for about half of a century, until 1841. House sizes were adjusted ahead of time to satisfy current political situation. To reiterate, Jefferson's method does not satisfy lower quota and it favors large states.

Webster's Method

In 1832 Daniel Webster entered his method in the list of candidates for fair apportionment. Jefferson's method was under-representing New England states, where Webster was from. Webster proposed that "…let the rule be, that the population of each state shall be divided by a common divisor, and, in addition to the number of members resulting from such division, a member shall be allowed to each state whose fraction exceeds a moiety of the divisor." Webster's method, also called method of major fractions is, like Jefferson method, based on the notion of picking the greatest divisor. Webster's method uses the divisor L(s) = a(s) + ½.. House size continued to be fixed and constant debates about mathematics and their influence on the representation of a particular state continued to be raised by congressmen. For example on the basis of census of 1900, the Alabama paradox became an evident problem. Here is what a representative of Maine had to say about it, after his state suffered a loss of a representative. "It does seem as though mathematics and science have combined to make a shuttlecock and battledore of the State of Maine…. God help Maine when mathematics reach for her!" Webster's method does not satisfy upper or lower quota, but it does not favor large or small states.

Huntington Method

In 1921 Edward V. Huntington, a Harvard mathematician came up with a method, known as method of Equal Proportions. He argued that because states vary so much in size and population, when ratio of their representatives to populations is compared, some of the states are shortchanged compared to other ones. He proposed to measure that shortchangedness, and transfer representatives if the transfer minimizes it. He proposed conducting the pair-wise comparisons until transfer of a representative does not minimize shortchangedness between any two states. The question is how to measure the shortchangedness? Huntington proposed comparing district sizes of two states and expressing the difference in terms of a fraction of the smaller district size. This is, however, only one way to express relative difference between two states. You may want to only compute the difference between two average district sizes and be done with it. The divisor Equal Proportions method uses is L(s) = $\sqrt{a(s)(a(s)+1)}$ A lot of methods can claim to be the best at measuring shortchangedness. But Huntington's method does not satisfy Quota. This flaw was painted over by Huntington, and solutions in the particular cases of that time did not seem so bad. But there is really no limit to how far from quota representation results using Equal Proportions method might get. By definition, Huntington method does not allow Alabama paradox. It also favors small states. Actually, all the methods based on the divisor criteria are Huntington methods, and all of them do not allow Alabama paradox. Two mathematicians, Balinski and Young, proved that there is no Huntington method that satisfies *quota* (being within one (exclusive) of the exact proportion); only the method of Smallest Divisors satisfies *upper quota* (ceiling of exact representation), and only Jefferson's method satisfies *lower quota* (floor of exact representation).

In November 1941 President Franklin D. Roosevelt signed "An Act to Provide for Apportioning Representatives in Congress among the several States by the equal proportions method". It was done for a political reason. In 1941 Arkansas received an extra seat, instead of Michigan. Arkansas was a Democratic state, while Michigan was Republican, and there were more Democrats in Congress at the time.

Quota Method

The question – Is there a method that does not allow the Alabama paradox and satisfies quota, was answered by Balinski and Young, who devised a method that was a refinement of the Huntington method. Instead of comparing all the states in the minimization of the shortchangedness, only states that are eligible to receive a seat or to lose a seat are considered. Eligibility means that they won't exceed upper quota or won't go below lower quota upon receiving or loosing a seat. But this method still favors large states, since Jefferson's method is used to compare the states. The Quota Method authors looked at the problems with Huntington Method and tried to apply the same technique in their method, while avoiding violation of Quota. However, their method still favors large states, and that is one of the reasons it is not used in the congress today. Another reason might be that it is too complicated for an average person to understand.

Balance Method

We discovered one additional method that could be used for the apportionment of representatives. The idea was to minimize the advantage of large states over small states in the Jefferson Method.

Terms

H – House
S – State population
U – US population
C – Coefficients to balance out the effect of large state

For each state this method uses the formula

$$\left\lfloor \frac{HS}{U\left(1+\dfrac{e}{1+\dfrac{S}{UC}}\right)} \right\rfloor$$

First, we calculate the exact number state should receive and then we multiply that number by proportion that is a somewhat bigger than one. Epsilon is a number between 0 and 1 that is balancing out the effect of truncation, but it introduces favoring large states. To reduce the effect of epsilon for large states and satisfy upper quota there is second number C, and it is picked to balance that effect and make sure that the results satisfy quota. This method satisfies Quota and it uses same formula for representation from all the states. However it does favor small states, and it does admit the Alabama paradox. In order to satisfy Quota an *ad hoc* technique is used, and that does not provide a stable solution. Compared to other methods the Balance method provides an alternative where someone would wish to give an advantage to smaller states and stay within Quota. This is also the reason for the results of the Balance method to be similar to the results of the method of Smallest Divisors, which also favors small states.

Conclusion

It is not possible to satisfy all of the requirements imposed by political needs. No known method satisfied all the requirements listed in the Rules for US Representatives apportionment section. The requirement, giving each state at least one representative, could be enforced in all of these methods but it would not change their properties. The political agenda more than anything else dictates the method used in Congress at any particular time. What may seem as an undesirable property from the mathematical point of view, may be admissible in politics, and vice versa.

References

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Hoffman, Paul. 1988. *Archimedes' Revenge The Joys and Perils of Mathematics.* New York: W.W. Norton & Company

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Results Table

STATE NAME	POP.	EXACT	H	JM	WM	EP	QM	BM	SD	HM
State 1	15475	1.454	2	1	1	2	1	2	2	2
State 2	35644	3.349	3	3	3	3	3	3	3	3
State 3	98756	9.279	9	10	10	9	10	9	9	9
State 4	88346	8.301	8	9	9	8	9	8	8	8
State 5	369	0.035	0	0	0	1	0	1	1	1
State 6	85663	8.049	8	8	8	8	8	8	8	8
State 7	43427	4.080	4	4	4	4	4	4	4	4
State 8	84311	7.922	8	8	8	8	8	7	7	7
State 9	54730	5.142	5	5	5	5	5	5	5	5
State 10	25467	2.393	3	2	2	2	2	3	3	3

H – Hamilton Method
JM – Jefferson Method
WM – Webster Method
EP – Huntington Method
QM – Quota Method
BM – Balance Method
SD – Smallest Divisors
HM – Harmonic Mean