Exploration of the three-person duel

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1 The duel

Picture a duel: two shooters facing one another, taking turns firing at one another, each with a fixed probability of hitting his opponent. The complete analysis of this two-variable problem is simple. Let's take a look.

Let the first shooter hit the second with probability p, and miss with probability $\overline{p}$. Of course we have $p + \overline{p} = 1$. Likewise, define q and $\overline{q}$. So the first shooter wins exactly when he lands the first shot. Thus, the probability that the first shooter wins is given by the following geometric sum:

$$p + \overline{pq}p + \overline{p}^2\overline{q}^2p + \overline{p}^3\overline{q}^3p + ... = \frac{p}{1 - \overline{pq}} \tag{1}$$

And the probability that the second shooter wins is given by:

$$\overline{p}q + \overline{p}^2\overline{q}q + \overline{p}^3\overline{q}^2q + \overline{p}^4\overline{q}^3q + ... = \frac{\overline{p}q}{1 - \overline{pq}} \tag{2}$$

It should reassure the reader that, since exactly one of the shooters must win the duel, the formulas do indeed add to 1.

Using these formulas, analysis is straightforward. Either player increases his chance of winning by increasing his chance of hitting. By satisfying

$$p = \frac{q}{1 + q} \tag{3}$$

we equalize each players probability of winning — that is, they each will win with probability 0.5.

2 The truel problem

Now that we can handle two shooters, what happens if we add a third? The picture gets more fuzzy. Is it even clear who to shoot?

To answer this and other questions, first we must specify some rules. Each member of this three-person duel — or truel — goes in a predefined order (player 1, then 2, then 3, and repeat as necessary) firing a shot at one of the others. Each is required to do his best to hit the opponent of his choice, rather than strategically fire into the air. The truel continues until one combatant remains standing. Say the probability that player 1 hits his target is p, with his chance of missing q_1 (again satisfying $p+\overline{p}=1$). Likewise we have q, $\overline{q}$, r, and $\overline{r}$, with all of these quantities universally known. Each participant is standing sufficiently far apart so that they cannot accidentally hit the wrong target. Now we can ask, and answer, who each player's target should be.

Consider the player currently attempting his shot. If he takes misses, no matter his target, the truel continues with the next player. If he shoots and hits his target, then he finds himself in a duel with the remaining player, with the latter going first. Since it is always preferable to duel someone with worst possible marksmanship, any player's natural target is whichever person has the best marksmanship. Thus, independent of who is aiming at a particular player, we see that his target is fixed. With this worked out, the following formulas and more can be generated for each players' chances at winning.

These apply whenever $p > q > r$:

$$P(1wins) = \left(\frac{p}{1-\overline{pqr}}\right)\left(\frac{\overline{r}p}{1-\overline{pr}}\right) \tag{4}$$

$$P(2wins) = \left(\frac{q}{1-\overline{pqr}}\right)\left(\frac{\overline{p}q\overline{r}+\overline{pq}r}{1-\overline{qr}}\right) \tag{5}$$

$$P(3wins) = \left(\frac{r}{1-\overline{pqr}}\right)\left(\frac{p}{1-\overline{pr}}+\frac{\overline{p}q}{1-\overline{qr}}+\frac{\overline{pq}^2r}{1-\overline{qr}}\right) \tag{6}$$

(All formulas may be found in the appendix)

3 Equilibrium solutions

From here, we can search for triples (p, q, r) such that each player will win a third of the time, similarly to (3). For instance, consider $(\frac{\sqrt{7}-1}{3}, 1, \frac{5-\sqrt{7}}{9})$, which is roughly (.55, 1, .26). Here, 1 will aim for 2, and hit 55% of the time, and will proceed to win the duel with 3 about 60% of the time, meaning 1 will win roughly 34% of the time. Meanwhile, if 1 misses (45% of the time), then 2 will surely hit him and proceed to beat 3 74% of the time, making 2 the winner 33% of the time. This leaves 33% of the victories to player 3. So roughly they work out close, and in fact the exact numbers work out perfectly. It is surprising that you can have an objectively fair truel in which one of the players never misses! There exist these so-called "equilibrium solutions" in each region of shooting patterns – that is, for $p > q > r$, $p > r > q$, etc. While it is not obvious that each region should have these solutions, the following points show that they do (some solutions are approximate). Note that (4), (5), and (6) apply only when $p > q > r$, so these cannot be used to verify all triples given.

$p > q > r$: (0.2761, 0.2760, 0.1270)
$p > r > q$: (0.250, 0.115, 0.242)
$q > p > r$: $(\frac{\sqrt{7}-1}{3}, 1, \frac{5-\sqrt{7}}{9})$

$q > r > q$: $(\frac{3-\sqrt{3}}{3}, 1, \frac{5+\sqrt{3}}{11})$
$r > p > r$: $(0.418, 0.250, 1)$
$r > q > p$: $(0.263, 0.386, 1)$

Now, looking to calculus, the Implicit Function Theorem reveals that, in fact, each of these points exists on a continuum of such solutions. This is intuitive when viewing the formulas as 2 equations in 3 variables (the third equation is forced by the necessity that all three must add to 1).

4 Graphical analysis

Although we have been able to find many equilibrium solutions, and have shown the existence of infinitely many, this still does not give a strong understanding of the dynamics of this problem. If I'm in a truel, with my fixed marksmanship, how good do I want my opponents to be? This is not at all clear. Of course if I'm much better than both opponents then I should be okay, but beyond that it's hard to say anything with certainty. While formulas help to bring precision to a problem, nothing beats pictures at showing whats really going on. I have created 3 sets of graphs, one set for each shooter in a truel. In each graph, we fix one marksmanship, and allow the other two to vary. Each color depicts which shooter has the highest chance of survival. Before looking at what these graphs show us, its worth making some notes on interpreting them. The graphs are available at `http://honors.cs.umd.edu/reports/truel/graphs.html`

Each graph is divided into regions of shooting patterns. For the graph of $\mathrm{P}(i \text{ hits}) = \rho$, the boundary lines are given by $y = x$, $y = \rho$, and $x = \rho$. In the bottom left of the graph, we get the case when i is the best shooter, with the line $y = x$ determining which of the other two shooters is next best. Although this region is navely though of as the best for i, it is important not to rule out the consequences of two slightly worse shooters teaming up against a common target.

In the top right of the graph, we find the region in which i is the worst shot of the 3, with $y = x$ serving the same purpose. Again we find that the nave expectation — that this is a bad position for i — is not always the case here. Instead, being the worst shooter allows i to guarantee his will not be the first to lose, and in fact will often have the first shot advantage in the resulting duel.

Finally, In the top left region of this graph, we see the results of when the first shooter is the second best shot. He therefore shoots at the best, but is shot at by the best as well. The third shooter also aims for the best. The bottom right of the graph is the same story, with the other two shooters roles reversed. These regions are not typically the best place for i, because he finds himself to be the target of the best shooter of the truel. However, the right conditions can make this set-up very comfortable.

So now that we know how to interpret them, what is there to learn from these graphs?

First, lets examine them with respect to equilibrium solutions — when each shooter has the same chance of survival. These solutions are actually very clearly presented in the graphs. They occur any time that seas of red, green, and blue in the same region all meet at a common point. Thus the first graph, P(1 hits) = 0.2, shows 6 equilibrium solutions — one in each region. In particular, it tells us there is an equilibrium near (0.1, 0.23, 0.18). Sure enough, the probabilities of survival at that point are roughly (0.34, 0.32, 0.34). This is a very simple way of ball-parking these solutions! Furthermore, looking at the progression of graphs for a given shooter, we can watch the movements of a certain family of equilibrium points as a given shooter increases/decreases his marksmanship. Although analytic methods show that such families of solutions must exist, it is rewarding to see them in these graphs.

Now, lets look the different regions of a particular shooters victories. For a given player, i, his graph shows potentially 6 regions of victory — one for each region of shooting patterns. The two in the bottom left of the graph (when i is the best shot) are always connected to one another, and lives in the bottom right of the region (when i is much better than the other two). This matches intuition, as two shooters slightly worse than i should be able to team up and beat i. However, if we look at this region for shooter 1, then 2, then 3, all at a fixed marksmanship level, well see the corresponding regions shrink from 1s domination down to 3s more worse showing. This is a good picture of the advantage of going earlier in the round. Shooter 1 wins most of the time when he is the strongest, but 3 loses most of the time, while 2 falls somewhere in the middle.

Similarly, we can restrict our interest only to the regions where i is the worst shooter. Here, we see shooter 1s survival-of-the-weakest region grow with his marksmanship, until it finally tops off and dies away into nothing by the time hes a .6 shot. Meanwhile, shooters 2 and 3 find their corresponding regions grow to fill the entire region! This may seem strange, but as I said before it actually makes good sense. As the worst shooter, they definitely make it to the duel when the first man dies. However, since theyll typically go first here, their 0.5-or-better shot makes them the favorite to win. So in this case, parallel situations give the first shooter an almost sure loss, and the second and third a probable win.

Finally, very different things occur in the other regions depending on position in the truel. Shooter 1 finds himself winning most truels in which he is the middle shot, though sufficiently strong opponents can always bring him down. Meanwhile, shooter 2 will almost never win when hes second to 1, but almost always when second to 3. Shooter 3 has the same trouble when a strong 1 fires at him, but can often beat a stronger 2.

5 Improvements

What are some improvements which could make these graphs more useful?

Instead of a winner-take-all coloring scheme, a more sophisticated graphics program could mix colors (50% blue, 30% red, and 20% green, for example) to give a stronger sense of how good each players chances are. This would turn all equilibrium points gray, and help give a cleaner picture of the continuity within a given graph.

Additionally, it would be very interesting to have enough of these graphs in a flipbook — say 100 for each shooter — to watch a clear, continuous animation of victory patterns. While it isn't *too* difficult to interpolate these graphs mentally, watching the rates of change could be enlightening. Coupling this with a continuum of colors would make for a very detailed, comprehensive, and attractive, presentation of this truckload of information.

6 Appendix: Formulas

We get a set of three formulas for each ordering of p, q, and r.

$p > q > r$:

$$P(1wins) = \left(\frac{p}{1-\overline{pqr}}\right)\left(\frac{\overline{r}p}{1-\overline{pr}}\right) \tag{7}$$

$$P(2wins) = \left(\frac{q}{1-\overline{pqr}}\right)\left(\frac{\overline{p}q\overline{r}+\overline{pq}r}{1-\overline{qr}}\right) \tag{8}$$

$$P(3wins) = \left(\frac{r}{1-\overline{pqr}}\right)\left(\frac{p}{1-\overline{pr}}+\frac{\overline{p}q}{1-\overline{qr}}+\frac{\overline{pq}^2r}{1-\overline{qr}}\right) \tag{9}$$

$p > r > q$:

$$P(1wins) = \left(\frac{p}{1-\overline{pqr}}\right)\left(\frac{\overline{q}p}{1-\overline{pq}}\right) \tag{10}$$

$$P(2wins) = \left(\frac{q}{1-\overline{pqr}}\right)\left(\frac{p}{1-\overline{pq}}+\frac{\overline{p}q\overline{r}}{1-\overline{qr}}+\frac{\overline{pq}r}{1-\overline{qr}}\right) \tag{11}$$

$$P(3wins) = \left(\frac{r}{1-\overline{pqr}}\right)\left(\frac{\overline{p}q+\overline{pq}^2r}{1-\overline{qr}}\right) \tag{12}$$

$q > p > r$:

$$P(1wins) = \left(\frac{p}{1-\overline{pqr}}\right)\left(\frac{p\overline{r}+\overline{pq}r}{1-\overline{pr}}\right) \tag{13}$$

$$P(2wins) = \left(\frac{q}{1-\overline{pqr}}\right)\left(\frac{\overline{p}q\overline{r}}{1-\overline{qr}}\right) \tag{14}$$

$$P(3wins) = \left(\frac{r}{1-\overline{pqr}}\right)\left(\frac{p}{1-\overline{pr}}+\frac{\overline{p}q}{1-\overline{qr}}+\frac{\overline{p}^2\overline{q}r}{1-\overline{pr}}\right) \tag{15}$$

$q > r > p$:

$$P(1wins) = \left(\frac{p}{1-\overline{pqr}}\right)\left(\frac{p\overline{r}}{1-\overline{pr}}+\frac{\overline{p}q}{1-\overline{pq}}+\frac{\overline{pq}r}{1-\overline{pr}}\right) \tag{16}$$

$$P(2wins) = \left(\frac{q}{1-\overline{pqr}}\right)\left(\frac{\overline{p}^2 q}{1-\overline{pq}}\right) \tag{17}$$

$$P(3wins) = \left(\frac{r}{1-\overline{pqr}}\right)\left(\frac{p+\overline{p}^2\overline{q}r}{1-\overline{pr}}\right) \tag{18}$$

$r > p > q$:

$$P(1wins) = \left(\frac{p}{1-\overline{pqr}}\right)\left(\frac{p\overline{q}+\overline{p}q}{1-\overline{pq}}\right) \tag{19}$$

$$P(2wins) = \left(\frac{q}{1-\overline{pqr}}\right)\left(\frac{p}{1-\overline{pq}}+\frac{\overline{p}^2 q\overline{r}}{1-\overline{pq}}+\frac{\overline{pq}r}{1-\overline{qr}}\right) \tag{20}$$

$$P(3wins) = \left(\frac{r}{1-\overline{pqr}}\right)\left(\frac{\overline{pq}^2 r}{1-\overline{qr}}\right) \tag{21}$$

$r > q > p$:

$$P(1wins) = \left(\frac{p}{1-\overline{pqr}}\right)\left(\frac{p\overline{q}}{1-\overline{pq}}+\frac{\overline{p}q}{1-\overline{pq}}+\frac{\overline{pq}r}{1-\overline{pr}}\right) \tag{22}$$

$$P(2wins) = \left(\frac{q}{1-\overline{pqr}}\right)\left(\frac{p+\overline{p}^2 q}{1-\overline{pq}}\right) \tag{23}$$

$$P(3wins) = \left(\frac{r}{1-\overline{pqr}}\right)\left(\frac{\overline{p}^2\overline{q}r}{1-\overline{pr}}\right) \tag{24}$$